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ON LAMINAR STEADY FLOW OF GAS MIXTURES IN PIPES AND CHANNELS

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The simplest problem of flow of a gas mixture between parallel plates or in a circular pipe is solved, using a new method developed by the author in [1], of solving a system of kinetic equations and reducing them to a system of hydrodynamic equations. A generalized Poiseuille's law is obtained for a flow of a binary gas mixture, and this enables us to reach a number of practically important conclusions.

Let us consider a steady motion of a binary gas mixture along the x-axis, between two planes separated from the coordinate origin by the distance of $y = \pm h$. In this case the system of hydrodynamic equations for a gas mixture can be written in the Navier-Stokes approximation [1] in the form

$$\frac{\partial^2 u_1}{dy^2} + \frac{d}{\mu_1}(u_2 - u_1) = \frac{1}{\mu_1}\frac{\partial P_1}{\partial x} \tag{1}$$

$$\frac{\partial^2 u_2}{dy_2} + \frac{d}{\mu_2}(u_1 - u_2) = \frac{1}{\mu_2}\frac{\partial P_2}{\partial x}$$

$$d = \frac{16}{3}\frac{\rho_1\rho_2}{m_1 + m_2}\,\Omega_{12}^{(1,1)} = \frac{n_1 n_2}{nD_{12}}kT$$

where u_1 and u_2 are the mean velocities of each component, D_{12} is the binary diffusion coefficient. The boundary conditions are $u_1(\pm h) = u_2(\pm h) = 0$. From (1) we readily obtain

$$\mu_1\frac{d^2u_1}{dy^2} + \mu_2\frac{d^2u_2}{dy^2} = \frac{\partial P_1}{\partial x} + \frac{\partial P_2}{\partial x} = \frac{\partial P}{\partial x} \tag{2}$$

Integrating this expression and using the boundary conditions, we obtain the following expressions for the mean velocity:

$$u_* = \frac{\mu_1 u_1 + \mu_2 u_2}{\mu_1 + \mu_2} = \frac{1}{2(\mu_1 + \mu_2)}\frac{\partial P}{\partial x}(y^2 - h^2)$$

It is clear that the mean velocity (and for the gas mixtures with equal kinematic viscosities this means the normal mass-averaged velocity), can be determined from the classical formula for the plane Poiseuille flow. Thus in the present case the mean velocity distribution in the gas mixture follows the usual Poiseuille law.

The motion and the velocity distribution of the separate components however, are subject to different laws which can be established by solving the system (1). With this purpose in mind, let us differentiate e.g. the first equation of (1) twice with respect to y. This yields

$$\frac{d^4u_1}{dy^4} + \frac{d}{\mu_1}\left(\frac{d^2u_2}{dy^2} - \frac{d^2u_1}{dy^2}\right) = 0$$

Using (2) we eliminate the derivative d^2u_2/dy^2 to obtain the following equations for u_1:

$$\frac{d^4u_1}{dy^4} - d\left(\frac{\mu_1 + \mu_2}{\mu_1\mu_2}\right)\frac{d^2u_1}{dy^2} = -\frac{d}{\mu_1\mu_2}\frac{\partial P}{\partial x}$$

The boundary conditions can be written in the form

$$u_1(\pm h) = 0, \qquad \left(\frac{d^2u_1}{dy^2}\right)_{y=\pm h} = \frac{1}{\mu_1}\frac{\partial P_1}{\partial x}$$

in which case we obtain the following expression for the velocity u_1:

$$u_1 = \frac{1}{2(\mu_1+\mu_2)}\frac{\partial P}{\partial x}(y^2 - h^2) + \frac{\omega_1}{k^2_0}\left(\frac{\mathrm{ch}k_0y}{\mathrm{ch}k_0h} - 1\right) \tag{3}$$

$$k_0^2 = d\left(\frac{\mu_1+\mu_2}{\mu_1\mu_2}\right) \qquad \omega_1 = \frac{1}{\mu_1}\frac{\partial P_1}{\partial x} - \frac{1}{(\mu_1+\mu_2)}\frac{\partial P}{\partial x}$$

In an entirely analogous manner we obtain the expression for u_2 corresponding to (3) with the indices 1 and 2 interchanged. Clearly, $\mu_1\omega_1 + \mu_2\omega_2 = 0$, therefore an expression for u_* follows naturally from the expressions for u_1 and u_2. Since μ_s are positive, ω_1 and ω_2 have opposite signs.

From the expressions for u_1 and u_2 it is clear that in a binary gas mixture the velocity distribution of each component assumes the form of a sum or a difference of two distinct distributions, namely, of the mean parabolic distribution, and a more complex distribution described in terms of a hyperbolic cosine.

The expressions for u_1 and u_2 can be used to calculate the rate of gas flow through the channel cross section

$$Q_s = \int_{-h}^{+h} v_s dy = -\frac{2h^3}{3}\left[\frac{1}{\mu_1+\mu_2}\frac{\partial P}{\partial x} - \frac{6\omega_s}{(k_0h)^3}(k_0h - \mathrm{th}k_0h)\right]$$

When k_0h are small, we obtain

$$Q_s = -\frac{2}{3}\frac{h^3}{\mu_s}\frac{\partial P_s}{\partial x}$$

We can see that in this case the rate of flow of each gas component is determined by its partial pressure and the coefficient of viscosity, and the behavior of each gas component is independent of each other. However, when k_0h are large, the amount and the velocity distribution of each component will not differ appreciably from the values corresponding to the classical Poiseuille's law of flow in a channel. Thus the character of the flow of a binary mixture in a channel depends on the parameter k_0h.

Let us now consider a motion of a binary gas mixture through a circular pipe of radius R_0, the axis of which coincides with the x-axis. In this case we have $v_x = v(r)$, $v_r = v_0 = 0$, and the corresponding system of equations [1] can be written in the form

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial v_1}{\partial r} + \frac{d}{\mu_1}(v_2 - v_1) = \frac{1}{\mu_1}\frac{\partial P_1}{\partial x} \tag{4}$$

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial v_2}{\partial r} + \frac{d}{\mu_2}(v_1 - v_2) = \frac{1}{\mu_2}\frac{\partial P_2}{\partial x}$$

from which we readily obtain

$$\mu_1\frac{d}{dr}r\frac{dv_1}{dr} + \mu_2\frac{d}{dr}r\frac{dv_2}{dr} = r\frac{\partial P}{\partial x} \tag{5}$$

Integrating twice with respect to r and utilizing the no-slip conditions at the wall when $r = R_0$, we obtain the following mean velocity:

$$v_* = \frac{\mu_1v_1+\mu_2v_2}{\mu_1+\mu_2} = \frac{1}{4(\mu_1+\mu_2)}\frac{\partial P}{\partial x}(r^2 - R_0^2) \tag{6}$$

We see that a certain mean velocity (for the gases with the same kinematic viscosity this is the usual mass-averaged velocity) can be found using the classical Poiseuille's formula. To find the velocity distribution for each individual gas component, we must solve the system (4) using the no-slip condition at the wall and the restriction in the range of velocities attainable inside the pipe. Differentiating the first equation of (4) twice and performing certain simple manipulations, we obtain

$$\frac{d}{dr} r \frac{d}{dr} \frac{1}{r} \frac{d}{dr} r \frac{dv_1}{dr} - \frac{d}{\mu_1} \left(\frac{d}{dr} r \frac{dv_2}{dr} - \frac{d}{dr} r \frac{dv_1}{dr} \right) = 0$$

Eliminating v_2 with the help of (5) we obtain

$$\frac{1}{r} \frac{d}{dr} r \frac{dz}{dr} - k_0^2 z = - \frac{d}{\mu_1\mu_2} \frac{\partial P}{\partial x}, \quad z = \frac{1}{r} \frac{d}{dr} r \frac{dv_1}{dr} \tag{7}$$

where k_0^2 is given by (3). The general, zero-bounded solution of (7), has the form

$$z = cI_0(k_0 r) + \frac{1}{\mu_1 + \mu_2} \frac{\partial P}{\partial x}$$

where $(I_0(x)$ is the Bessel function.

The first equation of (4) shows that at the pipe wall

$$z_{r=R_0} = \frac{1}{\mu_1} \frac{\partial P_1}{\partial x}$$

This condition enables us to determine the arbitrary constant, and we obtain

$$z = \omega_1 \frac{I_0(k_0 r)}{I_0(k_0 R_0)} + \frac{1}{\mu_1 + \mu_2} \frac{\partial P}{\partial x}$$

Multiplying this expression by r, integrating with respect to r from zero to r, multiplying the result by $1/r$ and again integrating with respect to r from r to R_0, we finally obtain

$$v_1 = \frac{1}{4(\mu_1 + \mu_2)} \frac{\partial P}{\partial x} (r^2 - R_0^2) + \frac{\omega_1}{k_0^2} \frac{I_0(k_0 r) - I_0(k_0 R_0)}{I_0(k_0 R_0)} \tag{8}$$

Repeating the above procedure for the second equation of (4) we obtain the expression for v_2 corresponding to (8) with the indices 1 and 2 interchanged.

From this we see that the velocity distributions for the separate gas components are determined by a law which is more complex than the Poiseuille's parabolic law. However, it can easily be shown that the mean velocity will still be determined by the parabolic law (6). Calculating the rate of gas flow through the cross section of the pipe, we obtain

$$Q_s = - \frac{\pi R_0^4}{8} \left\{ \frac{1}{\mu_1 + \mu_2} \frac{\partial P}{\partial x} + \frac{8\omega_s}{k_0^2 R_0^2} \left[1 - \frac{2}{k_0 R_0} \frac{I_1(k_0 R_0)}{I_0(k_0 R_0)} \right] \right\} \tag{9}$$

For small $k_0 R_0$ the expression (9) becomes the Poiseuille's formula, containing however the partial viscosity and pressure gradient relating to the given component of the gas mixture

$$Q_s = - \frac{\pi R_0^4}{8\mu_s} \frac{\partial P_s}{\partial x}$$

When $k_0 R_0 \gg 1$, we obtain an expression for the rate of gas flow identical to the classical Poiseuille's formula

$$Q_s = - \frac{\pi R_0^4}{8(\mu_1 + \mu_2)} \frac{\partial P}{\partial x}$$

and this expression is an exact corollary of (6).

From the expressions (3) and (8) we see that the gas flow velocities consist of two terms, a certain mean velocity which is the same for all gas components, and a supplementary velocity which has different signs for different components and averages to zero. It is therefore clear that the supplementary velocities are related to the flow diffusion rates and are

$$w_s = \frac{\omega_s}{k_0^2}\left\{\frac{\operatorname{ch} k_0 y}{\operatorname{ch} k_0 h} - 1\right\}$$

for the flows between parallel plates, and

$$w = \frac{\omega_s}{k_0^2}\left[\frac{I_0(k_0 r)}{I_0(k_0 R_0)} - 1\right]$$

for the flows in circular pipes.

The above velocities result from the longitudinal pressure gradients and satisfy the no-slip conditions at the pipe wall. The expressions given in [2] for the diffusion velocities are approximate, and can not generally satisfy the no-slip conditions.

From the expressions (8) and (9) for the circular pipe it is evident that the character of the flow will significantly depend on the dimensionless parameter $k_0 R_0$. The magnitude of this parameter can be expressed in terms of the flow parameters, in the following manner:

$$k_0^2 = \frac{n_1 n_2}{n_1 + n_2}\,\frac{(\mu_1 + \mu_2)}{\mu_1 \mu_2}\,\frac{kT}{D_{12}}$$

For the molecules which interact according to the laws governing elastic spheres, the coefficient of binary diffusion D_{12} can be written in the form

$$D_{12} = \frac{3}{8}\sqrt{\frac{\pi k T}{2 m_{12}}}\,\frac{1}{\pi n \sigma^2}$$

Simple manipulations then yield the following approximate expression:

$$(k_0 R_0)^2 \approx \frac{1 + \mu_2/\mu_1}{\left(1 + \frac{n_2}{n_1}\right)\sqrt{1 + m_2/m_1}}\,\frac{(2R_0)^2}{\lambda_2 \lambda_{12}}$$

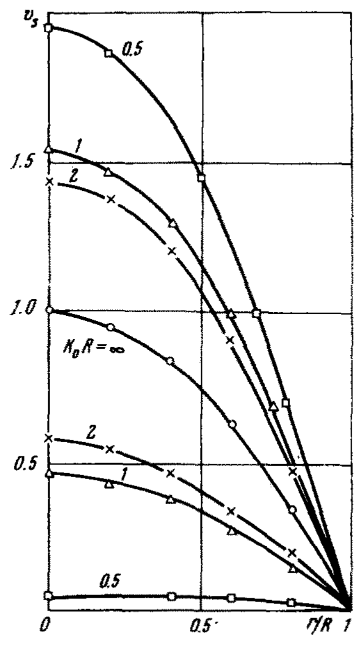

Fig. 1

We see that the parameter $k_0 R_0$ is inversely proportional to the free path length, i.e. to the Knudsen number. For this reason we have, in general, $k_0 R_0 \gg 1$ and the flow of a gas mixture will deviate little from the flow of a homogeneous gas. The difference will however be substantial when the values of the parameter become small. Figure 1 depicts the distribution of the relative velocities along the pipe radius in a binary gaseous mixture, for various values of the parameter $k_0 R_0$ (the conventional Poiseuille's law corresponds to the values $k_0 R \gg 1$). It was assumed that the partial pressure gradient was zero for one component and nonzero for the other component. It is evident that the component velocity distributions differ substantially from each other over a wide range of the parameter values. When $k_0 R_0 = {}^1/_2$ the velocities differ from each other by an order of magnitude.

The results obtained lead to a number of novel relationships and bring to light some interesting phenomena. For example, a gas moving under the action of partial pressure brings into motion a gas at rest which has no partial pressure. This is the case of a molecular ejector. The effectiveness of the performance of a molecular ejector can be determined using the theory expounded above. In a number of cases a gas can be set into motion which opposes its pressure gradient.

The theory developed here and the phenomena discovered play a major part in a number of practically important problems, and in particular in the problems of separating the gas and liquid mixtures by means of porous and semipermeable membranes.

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FRACTURE MECHANICS OF PIEZOELECTRIC MATERIALS. RECTILINEAR TUNNEL CRACK ON THE BOUNDARY WITH A CONDUCTOR

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A condition governing crack growth in a piezoelectric material is formulated, and the problem of tunnel crack development on the boundary between a piezoelectric ceramic and an elastic isotropic conductor is considered as an illustration. The stress components, displacements, electric field potential, displacement of the electric induction, and the magnitude of the critical load associated with crack growth are determined.

1. Fracture condition for piezoelectric media. The mechanical stress tensor components σ_{ij} in the static loading of a piezoelectric medium are functions of not only the geometric deformations but also of the electrical field.

Let us select the electrical field and the strain tensor components as independent variables, and let us represent the equation of the piezoelectric medium in crystal physics Cartesian x, y, z coordinates as follows [1]:

$$\sigma_{ij} = c_{ijkl}^E \varepsilon_{kl} - e_{ijk} E_k, \qquad D_i = e_{kli} \varepsilon_{kl} + \varepsilon_{ik}^s E_k \qquad (i, j, k, l = 1, 2, 3) \quad (1.1)$$

Here c_{ijkl}^E are the elastic moduli of the medium, e_{ijk} are the piezoelectric moduli, ε_{ik}^s are adiabatic dielectric constants of the medium, ε_{kl} are strain tensor components, σ_{ij} are stress tensor components, E_k are electrical field strength components, and D_i are the vector components of the electrical induction.

Neglecting volume forces and the Maxwell equations in the absence of free charges, the equilibrium equations of the medium are: